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ORIGINAL RESEARCH

Infinite-population approval voting: A proposal

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Abstract
In this study, we propose a new direction of research on the axiomatic analysis of approval voting, which is a common democratic decision method. Its novelty is to examine an infinite population setting, which includes an application to intergenerational problems. In particular, we assume that the set of the population is countably infinite. We provide several extensions of the method of approval voting for this setting. As our main result, axiomatic characterizations of the extensions are offered by revealing a direct link between approval voting and the Borda rule. The characterized methods are natural extensions of the standard approval voting method for the finite-population case and are regarded as minimum requirements for other possible infinite-population extensions, which are reasonably democratic.

Keywords Approval voting · Axiomatic analysis · Characterization · Infinite population · Intergenerational equity

1 Introduction

Approval voting is a common democratic method of collective decision-making, where each individual can write down any number of candidates or outcomes in the ballot, and then, the total number of approvals is derived for each outcome. Approval voting chooses outcomes that obtain the greatest number of approvals. This voting method was substantially developed by Brams and Fishburn (1978). Indeed, they demonstrate that approval voting is strategy-proof under dichotomous preferences, implying that

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approval voting is practically useful for resolving conflicts of interests among people. Since then, various extensions of approval voting have been proposed and examined.

Notably, the existing studies on approval voting methods focus on cases where the number of voters is finite. In this study, we examine how approval voting can be extended to an infinite-population setting. Specifically, we consider the case where the set of individuals is countably infinite. This extension provides a new perspective on voting problems. The rest of our study is organized as follows. Section 2 explains the relevance of collective decision-making with an infinite population. Section 3 reviews our methodology and results, and Section 4 illustrates related studies. After presenting a benchmark result in the finite-population framework in Sections 5 and 6 formulates the framework with an infinite population. Section 7 provides a direct extension of the benchmark result for the infinite-population setting. Section 8 shows three possible directions for resolving intergenerational conflicts; we especially emphasize the importance of limit concepts. Section 9 concludes.

2 Motivation of voting with an infinite population

In this section, we elaborate the significance of the infinite-population setting. In the real world, any voting procedure includes only a finite population. However, an infinite-population setting is normatively relevant for policy issues. There are many intergenerational conflicts including future generations. For instance, we can imagine the problem of just saving, climate change, national debts, social security, and so on.

Notably, there are finite-population models that are used for examining a trade-off between the current and future generations; it is indeed natural to consider a finite population if a short-term policy is considered. However, there are issues that make it plausible to consider an infinite population as a set of stakeholders. Indeed, long-term issues, such as saving and climate change, must be considered with an infinite-population.[1] We believe that the future is, by its fundamental nature, unbounded if one takes it as the number of periods the Earth has left. If we assume that the future is considered bounded, then there is a generation with no future. By contrast, if each generation t has its successive generation $t + 1$, the set of generations is necessarily infinite.

On the ground of this spirit, there are a considerable number of works that examine a trade-off among the current and future generations by taking an infinite population into account. One possible (and popular) approach to address these issues is applying a variant of utilitarian principles. Based on the classical work by Ramsey (1928) and von Weizsäcker (1965) proposes one of the most common criteria. His criterion requires that a utility stream is better than another if the former *overtakes* the latter: from some period onward, the cumulative sum of utilities is greater in the overtaking stream. Another common criterion is utilitarianism with discounting. That is, a policymaker applies weight in the form of exponential discounting to each generation.

[1] Rawls (1971) examines the problem of just saving in his theory of justice, employing a non-utilitarian approach. Rawls's argument is formalized by Arrow (1973) and Dasgputa (1974) by using a model with an infinite population.

A serious issue with this canonical utilitarian approach is how a policymaker can take utility levels of people in the future into account. Indeed, there are two main difficulties. First, it is not easy for the policymaker to know a utility level of each future generation. Second, more importantly, it is a difficult task for the policymaker to compare utility levels between the present and future generations. The second point is closely related to the historical development of the utility theory in economics; interpersonal comparability of utilities has been controversial among welfare economists; see, for example, Robbins (1938) and Pigou (1951). The problem of interpersonal comparison can be more serious in the intergenerational framework than in the intragenerational framework. Put differently, interpersonal comparison of utility levels among generations can be considered too demanding. By contrast, our framework only needs information about "which generation approves what". This point suggests that the requisite information for decision making is coarser. This is because it is natural to assume that future generations are likely to approve a policy if it provides a higher utility and disapprove the one that offers a lower utility. Then, given the status quo, one can expect whether or not the future generations approve, based on ordinal utilities. Here, we do not need cardinal utilities, and there is no need for interpersonal comparison. What we need to know/imagine is if the ordinal utility from a policy is higher than the utility level from the status quo. Thus, it is relatively not demanding to assume that the policymaker can access ballots of future generations.

Before moving to the explanations of our methodological approach, we highlight other possible interpretations of our infinite-population framework. First, assume that there is only one agent with different selves. Over the course of a lifetime, we assume that a person at time n is not the same as at time $n + 1$. If one asks this agent to opine regularly on a set of options, he or she can describe the evolution of his or her self according to a ballot profile. Consequently, this agent faces collective decision making with an infinite ballot stream.[2]

Given this point, a model of an infinite population can be plausible even with finite periods. We assume that people in the current period make a collective judgment. However, this judgment can affect people in the future in many respects. Indeed, their identities are consequences of the collective choice (Parfit 1984). Even if only a finite population actually appears in the next period, a set of possible states of the world, which includes people's identities as a part, can be infinite. From the viewpoint of the current period, each identity's interest is not ignorable. That is, there is an infinite set of an imaginable population even in the finite period.[3]

3 Methodological approach and main results

As mentioned in the previous section, an infinite-population setting is relevant for considering an intergenerational conflict, although using the cardinal utility levels of the future generations as an informational basis is difficult. Instead, we propose to use

[2] This interpretation is related to Mihara's (1997) argument, which provides a rationale for Arrovian social choice with an infinite population.

[3] McCarthy et al. (2019, p 2) provide similar rationales of an infinite-population framework of moral judgments.

information about their approvals, which are relatively imaginable from the viewpoint of the present generation. This implies that we use a ballot-voting model with an infinite population. Our proposal is to use an extension of the approval voting method as an aggregator. We explain why we focus on approval voting. Notably, the Borda rule is one of the most plausible voting method if people's personal rankings are available. It is axiologically consistent and robust, although it is not easy to obtain information about the personal rankings (hence, the Borda rule is rarely used in an actual political process). Approval voting shares its axiological relevance with the Borda rule since they are characterized by parallel axioms. Moreover, approval voting (and the Borda rule) can approximately achieve the utilitarian objective; see Pivato (2016). Given that the utilitarian approach is the prominent approach for resolving the intergenerational conflict, it is plausible to focus on approval voting.

Our methodological approach to the problem is axiomatic and can be divided into several steps. In the first step, we start from the axiomatic characterization of the approval voting function in the finite-population case, introduced by Fishburn (1978a,1978b) and refined by Alós-Ferrer (2006). This axiomatic result echoes Young's (1974) axiomatic characterization of the Borda rule. It combines three main axioms in a variable-electorate framework: faithfulness, consistency, and cancellation.

In the second step, we propose a method of adapting the principles embodied in these axioms within the framework of a fixed but countably infinite electorate.[4] For this, we divide the ballot profiles into two categories: (i) those for which only a finite set of voters are said to be concerned, and (ii) those for which an infinite number of voters said to be concerned. Here, a voter is said to be "unconcerned" if his or her ballot contains all the outcomes, that is, if he or she approves all outcomes; a voter is said to be "concerned" if he or she does not approve all outcomes. We establish new versions of faithfulness, consistency, and cancellation that apply to the set of ballot profiles in category (i). We then combine these axioms and show that they lead to the selection of the most approved outcomes on the set of profiles in this category.

In our last step, we offer three main directions to extend the above result to category (ii). The key is the definition of a stationarily approved outcome, an outcome that is always chosen by the finite-population approval voting from some period onward. Analogically, we define a stationarily disapproved outcome as a candidate that is never chosen by the finite-population approval voting from some period onward. Under the first extension, if each outcome is either stationarily approved or stationarily disapproved, then the outcome coincides with the set of approved outcomes. Under the second extension, any stationarily approved outcome is chosen whereas any stationarily disapproved outcome is not. The second is a refinement of the first. The final extension requires that an outcome is chosen if it is not stationarily disapproved. This is a further refinement. Interestingly, our extensions are restated by using the limit, limit superior, and limit inferior. For example, our final extension is equivalent to the limit superior of sequences that consist of outcomes under the finite approval voting. We note that the definitions of stationarily approved or disapproved outcomes have, in

[4] One may question whether our results are valid for the case where the set of individuals is uncountably infinite. The results in the second step are valid, but our results in the third step do not hold because our continuity axioms are constructed for the countable-set structure. If we impose an uncountable set of generations, we need related but other types of axioms.

a sense, similar forms to von Weizsäcker's overtaking criterion. For characterizations of these infinite-population extensions, we provide three axioms of choice continuity, which basically require that the social choice for an n-person society can be brought to that for infinite-population social choice if n is very large. To the best of our knowledge, our analysis is the first attempt to apply approval voting to an infinite-population setting.

4 Related literature

Fishburn (1978b) provides the first axiomatic characterization of approval voting. He employs a *ballot aggregation function*, which assigns outcomes to each profile, and shows that approval voting is the only ballot aggregation function that satisfies faithfulness, consistency, cancellation, and neutrality; see also Fishburn (1978a). Fishburn's characterization is substantially improved by Alós-Ferrer (2006) and Brandl and Peters (2019). In particular, Brandl and Peters (2019) unify two characterizations obtained by Fishburn (1978a,1978b).

In addition to the aforementioned works, several authors have conducted characterizations of approval voting; see (Sertel 1988; Baigent and Xu 1991; Goodin and List 2006, and Sato (2014). As mentioned above, Fishburn's axioms for approval voting are closely linked to those for the Borda rule; see Young (1974), Hansson and Sahlquist (1976), and Nitzan and Rubinstein (1981) for canonical characterizations of the Borda rule.[5] Indeed, the method of approval voting can be regarded as a type of scoring rules over dichotomous preferences; see Vorsatz (2007, 2008). This class of functions has been examined to analyze extended types of approval voting; see, for example, Alcalde-Unzu and Vorsatz (2009), Alcalde-Unzu and Vorsatz (2014) and Massó and Vorsatz (2008). All of these works on voting consider finite-population cases.[6]

Our characterization for the infinite-population case is related to the literature on intergenerational equity. To examine the overtaking criterion of von Weizsäcker (1965), Brock (1970) provides an axiom of robustness that is similar to our choice continuity axiom. Recently, many works have addressed characterizations of variants of the overtaking criterion systematically; see Asheim and Tungodden (2004) and Basu and Mitra (2007). They employ variants of Brock's axiom. However, all of them consider axioms for social preferences, while we consider continuity axiom for a social choice function. The main difference is that a social preference is a reflexive and transitive binary relation over the set of utility profiles, where the utility of each generation is a real number, while a social choice function assigns exactly one ballot to each ballot profile. Thus, the first case comprises ordering infinite sequences of real numbers interpreted as utility streams, while the second case examines how infinite sequences of ballots are aggregated in order to obtain a single ballot.

5 See also Smith (1973), Goertz and Maniquet (2011), and Pivato (2013) .

6 An exception is a work by Fey (2004), who examines infinite-population versions of the simple majority rule.

5 Formal argument for the finite-population case

Let $\mathbb{N} = \{1, 2, \ldots, \}$ be the set of natural numbers, and let N be any nonempty finite subset of $\mathbb{N}$. The set $\mathbb{N}$ represents the set of potential voters, and N is an **electorate**. Denote by $\mathcal{F}$ the collection of possible electorates, that is, the collection of all nonempty finite subsets of $\mathbb{N}$. Let X be a finite set of mutually exclusive outcomes. A **ballot** is any nonempty finite set B of X. Let $\mathcal{B} = 2^X \setminus \{\emptyset\}$ be the set of admissible ballots. Voters in N can cast any ballot in $\mathcal{B}$ but are unable to reveal their preferences among the outcomes in B. Each outcome in B is considered as **approved** and each outcome outside B is considered as **disapproved**. If a voter casts X, then he or she is considered as an **unconcerned voter**. Otherwise, he or she is considered as a **concerned voter**. A ballot profile $\mathbf{B}$, or simply a **profile**, on N is an ordered list of ballots $(B_i)_{i\in N} \in \mathcal{B}^N$, one ballot for each voter $i \in N$. Each order in which the ballots appear in the profile is feasible. Let

$$\Sigma = \bigcup_{N\in\mathcal{F}} \mathcal{B}^N$$

be the set of all possible profiles that one can construct from X and $\mathcal{F}$.

A profile $\mathbf{B} = (B_i)_{i\in N} \in \Sigma$ is N**-balanced** (or simply **balanced**) if

$$\forall x, y \in X, \quad \Pi(x, \mathbf{B}; N) = \Pi(y, \mathbf{B}; N),$$

that is, a profile is balanced if all outcomes received the same number of votes.

Finally, for any profile $\mathbf{B} = (B_i)_{i\in N} \in \Sigma$, define $M^N_{\mathbf{B}}$ as the subset of outcomes in X that receives the greatest number of approvals:

$$M^N_{\mathbf{B}} = \arg\max_{x\in X} \Pi(x, \mathbf{B}; N),$$

where

$$\Pi(x, \mathbf{B}; N) = |\{i \in N : x \in B_i\}|$$

represents the number of voters that approve the outcome $x \in X$ in $\mathbf{B}$.

A **ballot aggregation function** is a function $f : \Sigma \longrightarrow \mathcal{B}$ which assigns to every possible profile $\mathbf{B} \in \Sigma$ a nonempty set of outcomes (a ballot) in $\mathcal{B}$. Note that in this setting, initiated by Fishburn (1978a,1978b) and by Young (1974) in the context of scoring rules, the electorate is variable.

The **approval voting function** f^A on Σ is the ballot aggregation function defined as follows:

$$\forall \mathbf{B} = (B_i)_{i\in N} \in \Sigma, \quad f^A(\mathbf{B}) = M^N_{\mathbf{B}}.$$

Thus, the approval voting function chooses the nonempty subset of outcomes that obtain the greatest number of approvals among all outcomes.

We now provide three axioms for f based on the works by Alós-Ferrer (2006)and Brandl and Peters (2019). The first concerns electorates with a single voter.

Faithfulness: For each $B \in \mathcal{B}$, $f(B) = B$.

The second requires that whenever all outcomes receive exactly the same number of approvals, the ballot aggregation function selects the full set of outcomes.

Cancellation: For each balanced profile $\mathbf{B} \in \Sigma$, it holds that $f(\mathbf{B}) = X$.

For the last axiom, we define how to combine two profiles from disjoint electorates, let us say $N = \{i_1, \ldots, i_n\}$ and $N' = \{i'_1, \ldots, i'_{n'}\}$, where $N \cap N' = \emptyset$. For two profiles $\mathbf{B} = (B_{i_1}, \ldots, B_{i_n})$ and $\mathbf{B}' = (B'_{i'_1}, \ldots, B'_{i'_{n'}})$ in Σ, we associate the unique profile $\mathbf{B} + \mathbf{B}' \in \Sigma$ on the electorate $N \cup N'$ and defined as follows:

$$\mathbf{B} + \mathbf{B}' = (B_{i_1}, \ldots, B_{i_n}, B'_{i'_1}, \ldots B'_{i'_{n'}}).$$

The resulting profile $\mathbf{B} + \mathbf{B}'$ is the concatenation of the two profiles by placing ballots in $\mathbf{B}$ and ballots in $\mathbf{B}'$ besides each other so that they can be treated as one profile: the ballots $\mathbf{B}$ are placed first and then the ballots in $\mathbf{B}'$ are added. Note the binary operator $+$ is not commutative, that is, because the order matters here, $\mathbf{B} + \mathbf{B}'$ is not identical to $\mathbf{B}' + \mathbf{B}$. By contrast, it is associative, that is, for $\mathbf{B}$, $\mathbf{B}'$, and $\mathbf{B}''$ with pairwise disjoint electorates, $(\mathbf{B} + \mathbf{B}') + \mathbf{B}'' = \mathbf{B} + (\mathbf{B}' + \mathbf{B}'')$. To see these points, the following example is helpful.

Example 1 To fix the ideas, consider two disjoint electorates, say $N = \{1, 2, 3\}$ and $N' = \{5, 6, 7\}$. Each voter $i \in N \cup N'$ casts a ballot in $\mathcal{B}$. Consider two arbitrary profiles, one for each electorate, $\mathbf{B} = (B_1, B_2, B_3)$ and $\mathbf{B}' = (B'_5, B'_6, B'_7)$, respectively. Then, the resulting concatenated profile $\mathbf{B} + \mathbf{B}' = (B_1, B_2, B_3, B'_5, B'_6, B'_7)$ is different from the concatenated profile $\mathbf{B}' + \mathbf{B} = (B'_5, B'_6, B'_7, B_1, B_2, B_3)$. Next, consider a third electorate, say $N'' = \{9, 10\}$, and any profile $\mathbf{B}'' = (B''_9, B''_{10})$. On the one hand,

$$\begin{aligned}(\mathbf{B} + \mathbf{B}') + \mathbf{B}'' &= (B_1, B_2, B_3, B'_5, B'_6, B'_7) + (B''_9, B''_{10}) \\ &= (B_1, B_2, B_3, B'_5, B'_6, B'_7, B''_9, B''_{10}).\end{aligned}$$

On the other hand, $(\mathbf{B}' + \mathbf{B}'') = (B'_5, B'_6, B'_7) + (B''_9, B''_{10}) = (B'_5, B'_6, B'_7, B''_9, B''_{10})$, so that

$$\begin{aligned}\mathbf{B} + (\mathbf{B}' + \mathbf{B}'') &= (B_1, B_2, B_3) + (B'_5, B'_6, B'_7, B''_9, B''_{10}) \\ &= (B_1, B_2, B_3, B'_5, B'_6, B'_7, B''_9, B''_{10}).\end{aligned}$$

Therefore, we obtain $(\mathbf{B} + \mathbf{B}') + \mathbf{B}'' = \mathbf{B} + (\mathbf{B}' + \mathbf{B}'')$.

The formal statement of consistency is as follows.

Consistency: For each $\mathbf{B} = (B_i)_{i \in N}, \mathbf{B}' = (B'_i)_{i \in N'} \in \Sigma$ such that $N \cap N' = \emptyset$, if $f(\mathbf{B}) \cap f(\mathbf{B}') \neq \emptyset$, then $f(\mathbf{B} + \mathbf{B}') = f(\mathbf{B}) \cap f(\mathbf{B}')$.

The following theorem is a statement that is obtained from the conjunction of theorems by Alós-Ferrer (2006) and Brandl and Peters (2019).

Theorem 1 (Alós-Ferrer 2006; Brandl and Peters 2019 Theorem 1) *The approval voting function f^A is the unique ballot aggregation function satisfying faithfulness, consistency, and cancellation.*

Notably, Theorem 1 of Brandl and Peters (2019) states that an aggregation function is the approval voting method if and only if it satisfies disjoint equality[7], consistency, and faithfulness. Their result can be applied to the characterization by Fishburn (1978b) and Alós-Ferrer (2006); see Theorem 3 of Brandl and Peters (2019).

Theorem 1 shows a direct link between the characterization of the Borda rule and that of the approval voting method. Notably, Young's characterization of the Borda rule employs faithfulness, consistency, and cancellation without relying on anonymity (Young 1974). Since Theorem 1 does not rely on anonymity, the characterizations of the Borda rule and the approval voting method are completely parallel. The source of the difference between the two is the informational basis. Precisely, the Borda rule uses a personal ranking over outcomes or states, while the approval voting method uses a personally approved set. The approval framework can be regarded as a setting with a special class of rankings. If voter i casts B_i, one can see the options of B_i as preferred to those in $X \setminus B_i$, and options in B_i (resp. $X \setminus B_i$) as indifferent options for i. That is, B_i corresponds to a dichotomous ranking. Thus, the three axioms for approval voting are induced from those for the Borda rule, which employs axioms for any type of individual rankings.

6 Formal setting for the infinite-population case

Henceforth, we assume that N is **fixed**, countably infinite, and coincides with the set of natural numbers $\mathbb{N}$ endowed with the naturel order $\leq$. X is still assumed to be finite. In this setting, $\mathbb{N}$ represents the time structure corresponding to the natural order. Each $i \in N (= \mathbb{N})$ is the representative agent of people living in period i. That is, agent $i+1$ is the next generation of agent i. Each voter $i \in N$ submits his or her ballot $B_i \in \mathcal{B}$. An implicit assumption is that each generation's interest is aggregated by a certain procedure and is represented by B_i. A **profile B** is an element of $\mathcal{B}^{\mathbb{N}}$, that is, $\mathbf{B}$ is an ordered list of individual ballots $(B_i)_{i \in N}$. For each nonempty finite set of generations $M \subseteq N$, let

$$\Pi(x, \mathbf{B}; M) = |\{i \in M : x \in B_i\}|,$$

[7] This axiom indicates that if the electorate is composed of two voters approving disjoint sets of outcomes, then the ballot aggregation function selects the union of these two ballots.

which represents the number of voters in $M \subseteq N$ who approve the outcome x. A profile $\mathbf{B}$ is said to be **M-balanced** if

$$\forall x, y \in X, \quad \Pi(x, \mathbf{B}; M) = \Pi(y, \mathbf{B}; M).$$

In other words, in an M-balanced profile, each outcome obtains the same number of approvals in M. An **interval** I is a possibly infinite set of consecutive voters or generations. Formally, an interval I is defined as follows:

$$[i < j < k \wedge \{i, k\} \subseteq I] \implies [j \in I].$$

Each interval corresponds to a cluster of generations that lie next to each other. A **nonempty finite interval** I is of the form $\{i, i+1, i+2, \dots, j\} \subseteq \mathbb{N}$ and will be denoted by $[\![i, j]\!]$. The **support** of a profile $\mathbf{B}$ is the smallest interval I such that

$$\forall i \in N, \quad [B_i \neq X] \implies [i \in I].$$

The support of a profile represents the shortest range of time that can cover all concerned generations. In particular, the support of the profile $\mathbf{X} = (X, X, \dots, X, \dots)$, in which all agents are unconcerned, is empty. The subset of profiles with **finite support, possibly empty**, is denoted by Ω. A profile has finite support if the number of concerned individuals is finite. For notational convenience, we also use intervals included in $\mathbb{N} \cup \{0\}$.

A profile with finite support is a natural starting point for understanding the resolution of intergenerational conflicts. Even in the presence of an infinite set of agents in the discrete time flow, a conflict may exist among the agents in a finite set. Thus, profiles with finite support are treated similarly to profiles with a finite population. The following example helps capture the aforementioned concepts.

Example 2 Assume that the profile $\mathbf{B}$ is such that $B_1 = B_2 = B_5 = B_6 = X$, B_3, B_4, B_6, and B_7 differ from X, and $B_i = X$ for each $i > 7$. Then $\mathbf{B} \in \Omega$ and its support is the interval $[\![3, 7]\!]$. Note that the finite support $[\![3, 7]\!]$ of $\mathbf{B}$ contains the voters 5 and 6 such that $B_5 = B_6 = X$. □

Given a profile $\mathbf{B} \in \Omega$ with support $[\![i, j]\!]$, we define its **opposite profile** $-\mathbf{B} \in \Omega$ as follows: the support of $-\mathbf{B}$ is $[\![j+1, 2j-i+1]\!]$, and

$$\forall r \in [\![0, j-i]\!], \quad -B_{j+1+r} = \begin{cases} X \setminus B_{i+r} & \text{if } B_{i+r} \neq X, \\ X & \text{if } B_{i+r} = X. \end{cases}$$

Example 3 Consider the profile $\mathbf{B} \in \Omega$ of support $[\![3, 7]\!]$ given in Example 2. Its opposite profile $-\mathbf{B}$ whose support is $[\![8, 12]\!]$ is as follows: $-B_8 = X \setminus B_3$, $-B_9 = X \setminus B_4$, $-B_{10} = -B_{11} = X$, and $B_{12} = X \setminus B_7$. □

Given a profile $\mathbf{B} \in \Omega$ of support $[\![i, j]\!]$, its **t-translated profile $\mathbf{B}^t$**, where t is a positive integer, is a profile such that its support is $[\![i+t, j+t]\!]$ and

$$\forall r \in [\![0, j-i]\!], \quad B^t_{i+t+r} = B_{i+r}.$$

Example 4 Consider again the profile $\mathbf{B} \in \Omega$ of support $[\![3, 7]\!]$ given in Example 2. The support of its 2-translated profile is $[\![5, 9]\!]$, where $B^2_5 = B_3$, $B^2_6 = B_4, \ldots, B^2_9 = B_7$. □

A *generalized ballot aggregating function* is a function $f : \mathcal{B}^{\mathbb{N}} \longrightarrow \mathcal{B}$ that assigns to each profile $\mathbf{B} \in \mathcal{B}^{\mathbb{N}}$ a unique ballot $f(\mathbf{B}) \in \mathcal{B}$. Notably, most works on approval voting for the finite-population case allow the population to vary; see Section 5. In the current framework, the cardinality of the population is fixed, and the set N of voters is always the countably infinite set $\mathbb{N}$. A notable consequence is that the concatenation operator contained in consistency for the finite case can no longer be used. Indeed, the axiom of consistency considers the possibility to combine two profiles with a disjoint set of voters by concatenating them through the binary operator $+$. Thus, it is impossible to proceed in this way since the population of voters is now fixed. Therefore, we propose an alternative approach to combine two profiles of $\mathcal{B}^{\mathbb{N}}$ through the binary operator $\oplus : \mathcal{B}^{\mathbb{N}} \times \mathcal{B}^{\mathbb{N}} \longrightarrow \mathcal{B}^{\mathbb{N}}$. For any two profiles $\mathbf{B}$ and $\mathbf{B}'$ in $\mathcal{B}^{\mathbb{N}}$, the combined profile $\mathbf{B} \oplus \mathbf{B}'$ is defined as follows:

$$\forall i \in N, \quad (B \oplus B')_i = \begin{cases} B_i \cap B'_i & \text{if} \quad B_i \cap B'_i \neq \emptyset, \\ B_i \cup B'_i & \text{if} \quad B_i \cap B'_i = \emptyset. \end{cases} \tag{1}$$

An interpretation of this operator is that if this voter approves some outcomes in two profiles, then it suggests that this voter views these outcomes as safe outcomes. Consequently, the combined profile selects these commonly approved outcomes. If instead, this voter approves disjoint sets of outcomes in the two profiles, then this voter evidently has divergent opinions in the two profiles. Therefore, the combined profile selects the union of these two ballots, which expresses the voter's indecision to make a choice. However, no plausible reason exists for including outcomes outside of the union. For instance, imagine a situation where the same population has to opine on the outcomes of X from two divergent perspectives, $\mathbf{B}$ and $\mathbf{B}'$ reflecting two different criteria. If a voter approves the same outcome x in the two profiles $\mathbf{B}$ and $\mathbf{B}'$, then x is approved on both criteria and must be retained as a common approval in the combined profile; outcomes approved on only one of the two criteria may be considered comparatively less safe, and therefore, not retained in the combined profile. By contrast, if the voter has radically different opinions about the candidates concerning the two criteria, then his or her opinion in the combined profile remains ambivalent, and therefore, will be the union of its two ballots.

As the following point is employed later, we explicitly state it as a remark.

Remark 1 Notice that the binary operator $\oplus$ is commutative, that is, $\mathbf{B} \oplus \mathbf{B}' = \mathbf{B}' \oplus \mathbf{B}$, which is not the case for the binary operator of concatenation $+$ used in the finite case. In contrast to the operator of concatenation $+$, the binary operator $\oplus$ is not associative. In general, it is not true that $(\mathbf{B} \oplus \mathbf{B}') \oplus \mathbf{B}'' = \mathbf{B} \oplus (\mathbf{B}' \oplus \mathbf{B}'')$. Nevertheless, this equality is valid as long as $\mathbf{B}$, $\mathbf{B}'$, and $\mathbf{B}''$ have pairwise disjoint supports (weak associativity).

Despite differences between the operator $+$ on the elements of Σ and the operator $\oplus$ on the elements of $\mathcal{B}^{\mathbb{N}}$, the following remark draws a connection between these two operators.

Remark 2 From a finite electorate N and a profile $\mathbf{B} = (B_i)_{i \in N} \in \Sigma$, we can associate in a unique way the infinite profile $\overline{\mathbf{B}} \in \Omega \subseteq \mathcal{B}^{\mathbb{N}}$ as follows: for $i \in N$, $\overline{B}_i = B_i$ and $\overline{B}_i = X$ if $i \in \mathbb{N} \setminus N$. Note that the order chosen for the elements of N does not matter. To see this, consider the set $N = \{1, 3, 4\}$ and the profiles $\mathbf{B} = (B_1, B_3, B_4)$ and $\mathbf{B}^0 = (B_4, B_1, B_3)$ in Σ. By definition,

$$\overline{\mathbf{B}} = (B_1, X, B_3, B_4, X, \ldots, X, \ldots) = \overline{\mathbf{B}^0}.$$

Next, consider two finite profiles $\mathbf{B}$ and $\mathbf{B}'$ in Σ with disjoint supports N and N' respectively. It holds that

$$\overline{\mathbf{B} + \mathbf{B}'} = \overline{\mathbf{B}} \oplus \overline{\mathbf{B}'}. \tag{2}$$

In particular, (2) implies that

$$\overline{\mathbf{B} + \mathbf{B}'} = \overline{\mathbf{B}} \oplus \overline{\mathbf{B}'} = \overline{\mathbf{B}'} \oplus \overline{\mathbf{B}} = \overline{\mathbf{B}' + \mathbf{B}},$$

where the second equality follows from the fact that $\oplus$ is commutative. To see how equality (2) works on an example, consider the two finite profiles $\mathbf{B} = (B_1, B_3, B_4)$ and $\mathbf{B}' = (B'_2, B'_5, B'_7)$ in Σ. On the one hand, we have

$$\mathbf{B} + \mathbf{B}' = (B_1, B_3, B_4, B'_2, B'_5, B'_7) \text{ and } \overline{\mathbf{B} + \mathbf{B}'} = (B_1, B'_2, B_3, B_4, B'_5, X, B'_7, X, \ldots, X, \ldots).$$

On the other hand,

$$\overline{\mathbf{B}} = (B_1, X, B_3, B_4, X, \ldots, X, \ldots) \text{ and } \overline{\mathbf{B}'} = (X, B'_2, X, X, B'_5, X, B'_7, X, \ldots, X, \ldots).$$

Thus, by definition (1) of the operator $\oplus$, we obtain the desired result (2):

$$\overline{\mathbf{B}} \oplus \overline{\mathbf{B}'} = (B_1, B'_2, B_3, B_4, , B'_5, X, B'_7, X, \ldots X, \ldots) = \overline{\mathbf{B} + \mathbf{B}'}.$$

Before moving to our axiomatic analysis for the infinite-population case, we provide an additional comment on our framework. Notably, we use a model with multiple profiles, and thus, consider changes of ballot profiles. Indeed, the binary operator $\oplus$ uses two different profiles. Compared to a finite-population framework, a change in a profile may sound eccentric to our infinite-population framework. Our rationales are as follows. First, our aim is not to derive an actual policy recommendation but rather to consider a process for policy recommendations. Since how to respond to changes in people's ideas is crucial for examining the process of creating policy recommendations, an imaginable change in ballots plays an important role in our study. In other words, even if there is no actual change in the observed ballot profile, it is possible to "imagine" multiple profiles that could have arisen. Second, even if we focus only on actual changes in ballots, the time flow makes us consider different ballot profiles. To see this point, suppose that, in period t, a policymaker faces the following profile:

$$B_t, B_{t+1}, B_{t+2}, B_{t+3}, \ldots$$

Under this situation, generation t is the present generation. However, in period $t+1$, the policymaker faces the following profile, which is different from the previous one:

$$B_{t+1}, B_{t+2}, B_{t+3}, B_{t+4}, \ldots$$

Notably, generation $t+1$ is the present generation in this profile since generation t is now gone. Starting from the idea to apply the same aggregation procedure to both profiles, the decision-making process must necessarily treat different profiles. As such, we can incorporate different profiles of the future generations more realistically.

7 Three basic axioms for profiles with a finite support

This section extends the three basic axioms that appear in Theorem 1 from the finite-population setting to the subsets of profiles with finite supports Ω in the infinite-population setting. In the finite-population case, faithfulness requires that when there is only one voter, the outcome respects his or her ballot. In the current framework, such a situation does not belong to the domain of f. Consider the profile $\mathbf{X} = (X, X, X, X, \ldots)$, where each voter $i \in N$ approves each outcome. In this specific profile, no one has any strict concern about the outcome. If instead, we consider the profile $(B, \mathbf{X})$ in which only the first voter or generation possibly has specific concerns for some outcomes, then it is natural that the selected outcome should coincide with the new ballot B. Consequently, we reformulate faithfulness as follows.

Finite faithfulness: For each $B \in \mathcal{B}$, $f(B, \mathbf{X}) = B$.

Note that finite faithfulness implies $f(\mathbf{X}) = X$ by setting $B = X$. Without loss of generality, we could have defined finite faithfulness by saying that $f(\mathbf{B}) = B$ for any profile $\mathbf{B}$ with singleton support, and $f(\mathbf{B}) = X$ whenever the support of $\mathbf{B}$ is empty.

Subsequently, we provide a version of the principle of consistency when two profiles to be combined have disjoint and finite supports. This requires that if one combines two profiles where, in each profile, only a finite number of voters have specific concerns and these two subsets of voters are disjoint, then the principle of consistency applies. This is formally stated as follows.

Finite consistency: For each pair of profiles $\mathbf{B}, \mathbf{B}' \in \Omega$ with disjoint supports, the following holds. If $f(\mathbf{B}) \cap f(\mathbf{B}') \neq \emptyset$, then $f(\mathbf{B} \oplus \mathbf{B}') = f(\mathbf{B}) \cap f(\mathbf{B}')$.

Example 5 Consider the profile $\mathbf{B} \in \Omega$ of support $[\![3, 7]\!]$ illustrated in Example 2 and the profile $\mathbf{B}' \in \Omega$ with support $[\![9, 11]\!]$. Further, assume that the outcome $x \in f(\mathbf{B}) \cap f(\mathbf{B}')$. Note that the combined profile $\mathbf{B} \oplus \mathbf{B}'$ belongs to Ω and its support is the finite interval $[\![3, 11]\!]$. In particular, by definition of the binary operation $\oplus$ given in (1), we have $(B \oplus B')_i = B_i$ for $i \in [\![3, 7]\!]$ and $(B \oplus B')_i = B'_i$ for $i \in [\![9, 11]\!]$. In this case, if we combine two disjoint subsets of voters $[\![3, 7]\!]$ and $[\![9, 11]\!]$, both of which contain voters with specific concerns, then we may expect that the outcome x will continue to be selected if we combined these two profiles, that is, $x \in f(\mathbf{B}) \cap f(\mathbf{B}')$. Moreover, we may also expect that no other outcome outside $f(\mathbf{B}) \cap f(\mathbf{B}')$ should be selected by the rule for the combined profile $\mathbf{B} \oplus \mathbf{B}'$ because the only added voter in the support $[\![3, 11]\!]$, compared to the supports $[\![3, 7]\!]$ and $[\![9, 11]\!]$, is the voter 8 who casts the ballot $B_8 = X$, suggesting that he or she has no strict concern about the outcome. □

The last axiom requires that the principle contained in the axiom of cancellation for the finite setting is applied to the set of profiles with a finite support.

Finite cancellation: For each profile $\mathbf{B} \in \Omega$ of support I, if $\mathbf{B}$ is I-balanced, then $f(\mathbf{B}) = X$.

We have the material to study the consequences of the above three axioms applied to f on Ω. First, the combination formed by finite consistency and finite cancellation implies that the only thing that matters for f is the number of approvals for each voters. Put differently, finite consistency and finite cancellation imply a principle of finite anonymity on Ω.

Lemma 1 *Assume that a generalized ballot aggregating function f satisfies finite consistency and finite cancellation. For all profiles $\mathbf{B}, \mathbf{B}'$ in Ω, if the supports of $\mathbf{B}$ and $\mathbf{B}'$ are I and I', respectively, and*

$$\forall x \in X, \quad \Pi(x, \mathbf{B}; I) = \Pi(x, \mathbf{B}'; I'), \tag{3}$$

then $f(\mathbf{B}) = f(\mathbf{B}')$.

Proof Assume first that at least one of the two supports, I and I', is empty, say $I = \emptyset$. In such a case, $\mathbf{B} = \mathbf{X}$, and we have:

$$\forall x \in X, \quad \Pi(x, \mathbf{B}; \emptyset) = 0, \text{ and thus, by condition (3)}, \ \forall x \in X, \quad \Pi(x, \mathbf{B}'; I') = 0,$$

which forces $\mathbf{B}' = \mathbf{X}$. Therefore, $f(\mathbf{B}) = f(\mathbf{B}')$, as asserted.

Next, suppose that both I and I' have nonempty support. Let $I = [\![i, j]\!]$ be the support of $\mathbf{B}$ and let $I' = [\![i', j']\!]$ be the support of $\mathbf{B}'$. We proceed in two steps. In the first step, assume that $2j - i + 1 \leq i' - 1$. Let $-\mathbf{B}$ be the opposite of $\mathbf{B}$, whose support J is $[\![j + 1, 2j - i + 1]\!]$. The above inequality implies that the supports I, I', and J are pairwise disjoints. The profile $\mathbf{B} \oplus (-\mathbf{B}) \in \Omega$ and is $[\![i, 2j - i + 1]\!]$-balanced. Similarly, $(-\mathbf{B}) \oplus \mathbf{B}'$ is a $[\![j + 1, j']\!]$-balance profile. By finite cancellation, we have

$$f((-\mathbf{B}) \oplus \mathbf{B}') = f(\mathbf{B} \oplus (-\mathbf{B})) = X. \tag{4}$$

From the above equality, we deduce that

$$f(\mathbf{B}) = f(\mathbf{B}) \cap X = f(\mathbf{B}) \cap f((-\mathbf{B}) \oplus \mathbf{B}').$$

The support $[\![j+1, j']\!]$ of $-\mathbf{B} \oplus \mathbf{B}'$ has an empty intersection with the support I of $\mathbf{B}$. Thus, by finite consistency, we obtain the following:

$$f(\mathbf{B}) \cap f((-\mathbf{B}) \oplus \mathbf{B}') = f(\mathbf{B} \oplus ((-\mathbf{B}) \oplus \mathbf{B}')).$$

By Remark 1 (weak associativity), we have

$$f(\mathbf{B} \oplus ((-\mathbf{B}) \oplus \mathbf{B}')) = f((\mathbf{B} \oplus (-\mathbf{B})) \oplus \mathbf{B}'),$$

and thus,

$$f(\mathbf{B}) = f((\mathbf{B} \oplus (-\mathbf{B})) \oplus \mathbf{B}'). \tag{5}$$

By contrast, the support of $\mathbf{B} \oplus (-\mathbf{B})$ is $[\![i, 2j-i+1]\!]$, and by assumption, has an empty intersection with I'. The conjunction of (4) and finite consistency implies the following:

$$f(\mathbf{B}') = X \cap f(\mathbf{B}') = f(\mathbf{B} \oplus (-\mathbf{B})) \cap f(\mathbf{B}') = f((\mathbf{B} \oplus (-\mathbf{B})) \oplus \mathbf{B}'). \tag{6}$$

From (5)–(6), we conclude that $f(\mathbf{B}) = f(\mathbf{B}')$, as desired.

In the second step, assume that $2j - i + 1 > i' - 1$. We then use a translated profile $(\mathbf{B}')^t$. For a sufficiently large t, the proof with the assumed inequality holds for the pairs $(\mathbf{B}, (\mathbf{B}')^t)$ and $(\mathbf{B}', (\mathbf{B}')^t)$, from which we get the result for the pair $(\mathbf{B}, \mathbf{B}')$. This completes the proof of the lemma. □

This lemma is a powerful implication from the viewpoint of intergenerational justice. Since the anonymity requirement is implied by finite consistency and finite cancellation, the present generation and any generation in the future must be treated equally. In other words, there must be no discounting in the future. This point becomes clearer when we consider a translated profile. That is, for any profile, the choice is invariant for any t-translation; if $\mathbf{B}' = \mathbf{B}^t$ in the statement of Lemma 1, we obtain that $f(\mathbf{B}') = f(\mathbf{B}^t)$. This can be called the time-neutrality property. This idea is supported by Sidgwick (1907, p. 414), who writes "the time at which a man exists cannot affect the value of his happiness from a universal point of view".[8]

Corollary 1 *Assume that f satisfies finite consistency and finite cancellation. Then, for each profile $\mathbf{B}$ of Ω, and each positive integer t, it holds that $f(\mathbf{B}) = f(\mathbf{B}^t)$.*

The following result characterizes the implication of the three basic axioms, finite faithfulness, finite consistency, and finite cancellation. It indicates that such a generalized ballot aggregating function coincides with an approval method on the subdomain of profiles with finite supports. Precisely, for each $\mathbf{B} \in \Omega$ with (finite) support I, define $M^I_{\mathbf{B}}$ as follows:

$$M^I_{\mathbf{B}} = \arg\max_{x \in X} \Pi(x, \mathbf{B}; I),$$

[8] Ramsey (1928) follows Sidgwick's proposal of time neutrality by taking suggestions of J.M. Keynes.

which represents the subset of outcomes that receive the greatest number of approvals in I. In case $I = \emptyset$, that is, $\mathbf{B} = \mathbf{X}$, then $\Pi(x, \mathbf{B}; \emptyset) = 0$ for each $x \in X$, and thus, $M^I_{\mathbf{B}} = X$.

Theorem 2 *A generalized ballot aggregating function f satisfies finite faithfulness, finite consistency, and finite cancellation if and only if*

$$\forall \mathbf{B} \in \Omega,\ f(\mathbf{B}) = M^I_{\mathbf{B}},$$

where I denotes the support of $\mathbf{B}$.

Theorem 2 relies on an intermediary result, which makes the connection between the subdomain $\Omega \subseteq \mathcal{B}^{\mathbb{N}}$ associated with the infinite-population setting and the domain Σ associated with the finite-population setting. For each non-negative integer $t \geq 0$, let $s_t : \Sigma \longrightarrow \Omega$ be the canonical surjection defined as follows: for each $\mathbf{B} = (B_1, \ldots, B_n) \in \Sigma$, $(s_t(\mathbf{B}))_{r+t} = B_r$ for each $r \in [\![1, n]\!]$, and $(s_t(\mathbf{B}))_i = X$ in any other case. By Corollary 1, $f(s_t(\mathbf{B}))$ does not depend on t.

Example 6 To understand how the mapping $s_t : \Sigma \longrightarrow \Omega$ works, consider the following instance: $t = 2$, $n = 4$, and $\mathbf{B} = (B_1, B_2, B_3, B_4) = (\{x^1\}, \{x^1, x^2\}, \{x^3\}, \{x^3, x^5\}) \in \Sigma$. Then, for each $r \in [\![1, 4]\!]$, $(s_2(\mathbf{B}))_{r+2} = B_r$, and for each $i \in \{1, 2\} \cup \{7, 8, \ldots\}$, $(s_t(\mathbf{B}))_i = X$, so that

$$s_2(\mathbf{B}) = (X, X, \{x^1\}, \{x^1, x^2\}, \{x^3\}, \{x^3, x^5\}, X, \ldots, X, \ldots) \in \Omega.$$

□

Lemma 2 *If a generalized ballot aggregating function f satisfies finite faithfulness, finite consistency, and finite cancellation, then $f \circ s_0 = f^A$.*

Proof We prove that $f \circ s_0$ satisfies faithfulness, cancellation and consistency on Σ. Each axiom is shown to be satisfied in each of the following points:

- for each $B \in \mathcal{B}$, by finite faithfulness of f on $\mathcal{B}^{\mathbb{N}}$, $(f \circ s_0)(B) = f(B, \mathbf{X}) = B$. Thus, $f \circ s_0$ satisfies faithfulness on Σ;
- let $\mathbf{B} = (B_1, B_2, \ldots, B_n)$ be a balanced profile of Σ. It follows that $s_0(\mathbf{B}) = (\mathbf{B}, \mathbf{X})$ is a I-balanced profile of Ω, where I denotes the support of $s_0(\mathbf{B})$, where $I \subseteq [\![1, n]\!]$. Therefore, by finite cancellation of f, we have

$$(f \circ s_0)(\mathbf{B}) = X,$$

  which ensures that $f \circ s_0$ satisfies cancellation on Σ;
- let $\mathbf{B} = (B_1, B_2, \ldots, B_n)$ and $\mathbf{B}' = (B'_1, B'_2, \ldots, B'_{n'})$ be two profiles of Σ such that $(f \circ s_0)(\mathbf{B}) \cap (f \circ s_0)(\mathbf{B}') \neq \emptyset$. As $f \circ s_t$ does not depend on t, we have $(f \circ s_0)(\mathbf{B}') = (f \circ s_n)(\mathbf{B}')$ and thus, $(f \circ s_0)(\mathbf{B}) \cap (f \circ s_n)(\mathbf{B}') \neq \emptyset$. Note that $s_0(\mathbf{B})$ and $s_n(\mathbf{B}')$ have disjoint supports. Therefore, we can apply finite consistency to obtain:

$$f(s_0(\mathbf{B})) \cap f(s_n(\mathbf{B}')) = f(s_0(\mathbf{B}) \oplus s_n(\mathbf{B}')). \tag{7}$$

To show that $f \circ s_0$ satisfies consistency on Σ, we rewrite the right-hand side $f(s_0(\mathbf{B}) \oplus s_n(\mathbf{B}'))$ of (7) 'as follows. On the one hand,

$$s_0(\mathbf{B}) = (B_1, \dots, B_n, X, \dots, X, \dots) \text{ and } s_n(\mathbf{B}') = (\underbrace{X, \dots, X}_{n\, times}, B'_1, \dots, B'_{n'}, X, \dots, X, \dots).$$

By definition of the binary operator $\oplus$,

$$s_0(\mathbf{B}) \oplus s_n(\mathbf{B}') = (B_1, \dots, B_n, B'_1, \dots, B'_{n'}, X, \dots, X, \dots).$$

On the other hand,[9]

$$\mathbf{B} + \mathbf{B}' = (B_1, \dots, B_n, B'_1, \dots, B'_{n'})$$

and thus

$$s_0(\mathbf{B} + \mathbf{B}') = (B_1, \dots, B_n, B'_1, \dots, B'_{n'}, X, \dots, X, \dots)$$

Therefore, the right-hand side of (7) can be rewritten as

$$f(s_0(\mathbf{B}) \oplus s_n(\mathbf{B}')) = (f \circ s_0)(\mathbf{B} + \mathbf{B}'),$$

and thus

$$f(s_0(\mathbf{B})) \cap f(s_n(\mathbf{B}')) = ((f \circ s_0)(\mathbf{B})) \cap ((f \circ s_n)(\mathbf{B}')) = (f \circ s_0)(\mathbf{B} + \mathbf{B}').$$

Finally, because $f \circ s_t$ does not depend on t, we obtain

$$((f \circ s_0)(\mathbf{B})) \cap ((f \circ s_0)(\mathbf{B}')) = (f \circ s_0)(\mathbf{B} + \mathbf{B}'),$$

which shows that $f \circ s_0$ satisfies consistency on Σ.

Therefore, the problem is reduced to the finite-population case. By Theorem 1, one concludes that $f \circ s_0 = f^A$. □

We now have the material to prove Theorem 2.

Proof of Theorem 2 Let $n \in \mathbb{N}$ and $\pi : \Omega \longrightarrow \Sigma$ be the canonical function defined as follows: for $\mathbf{B} \in \Omega$ of support $I = [i, j]$, $\pi(\mathbf{B})$ is the element of Σ whose population size is $(j - i + 1)$ and such that, for $1 \leq r \leq j - i + 1$, $(\pi(\mathbf{B}))_r = B_{i+r}$. Note that $(s_{i-1} \circ \pi)(\mathbf{B}) = \mathbf{B}$, so that

$$f(\mathbf{B}) = (f \circ s_{i-1} \circ \pi)(\mathbf{B}) = (f \circ s_{i-1})(\pi(\mathbf{B})) = (f \circ s_0)(\pi(\mathbf{B})),$$

[9] See Section 4 for the definition of the operator $+$ on Σ.

since $f \circ s_{i-1} = f \circ s_0$ by Corollary 1. Thus, by Lemma 2,

$$f(\mathbf{B}) = f^A(\pi(\mathbf{B})) = M_{\pi(\mathbf{B})}^{[\![1,(j-i+1)]\!]} = M_{\mathbf{B}}^I,$$

which is the expected result. □

Notably, if a generation casts X, this generation is considered as a non-existent voter. Then, for the domain considered in Theorem 2, only a finite set of generations is considered as active. Theorem 2 states that collective decision-making follows from the standard approval voting over this finite-population active-voter set if one imposes the three axioms, which are extensions of the ones for the Borda rule in the model with a variable population.[10]

In sum, an aggregation rule must coincide with the finite-population approval voting method under the three basic axioms. Our result also states that the converse is also true. As long as an aggregation rule reflects the finite-population approval voting method, the three axioms are satisfied. The meaning of this converse implication is noteworthy and suggests that these axioms do not yield any restriction outside of Ω, in which only finitely many voters are concerned. Then, we cannot expect any further comparison over profiles with infinitely many concerned voters. However, most serious intergenerational conflicts include infinitely many concerned people in the future. This leads us to the next step.

8 Incorporating interests of future generations

In this section, we examine collective decision problems with substantial intergenerational conflicts by introducing three possible extensions of the approval voting function to deal with profiles with infinitely many concerned voters. Each of them satisfies different principles of continuity. Beforehand, we need to introduce three standard concepts for a sequence of sets: the **limit**, the **limit superior**, and the **limit inferior** of a sequence of sets of outcomes $(A^n)_{n\in\mathbb{N}}$, where $A^n \subseteq X$ for each n. In our setting of approval voting, these concepts represent the interests of generations in the distant future. First, the limit of $(A^n)_{n\in\mathbb{N}}$ is defined as follows:

$$\left[\lim_n A^n = A\right] \Longleftrightarrow \left[\exists m \in \mathbb{N} : \forall n \geq m,\, A^n = A\right],$$

that is, there is an integer m from which the sequence is constant. In such a case, we say that $(A^n)_{n\in\mathbb{N}}$ is a **convergent sequence**. Here, m can be a very large number, which represents a distant future. The limit inferior and the limit superior are defined as follows, respectively:

[10] It is noteworthy that, in a sense, Theorem 2 is a type of generalization of a characterization result for a variable, finite-population setting. We can identify a finite-population set of concerned generations as active voters in each profile in Ω, and, by regarding unconcerned generations in the distant future to be non-existent, we can induce the finite-population structure. Our result suggests that our aggregation rule(s) must choose the same outcome as the approval voting method under this finite-population model.

$$\lim_m \sup_{n\geq m} A^n = \lim_m \bigcup_{n\geq m} A^n = \bigcap_m \left(\bigcup_{n\geq m} A^n \right) \text{ and}$$
$$\lim_m \inf_{n\geq m} A^n = \lim_m \bigcap_{n\geq m} A^n = \bigcup_m \left(\bigcap_{n\geq m} A^n \right).$$

In other words, $x \in \lim_m \sup_{n\geq m} A^n$ if and only if

$$\forall m \in \mathbb{N}, \quad x \in \bigcup_{n\geq m} A^n,$$

indicating that regardless of how large m is, there exists $n \geq m$ such that $x \in A^n$. That is, x is in the limit superior if and only if x is in infinitely many A^n. If X is finite, the limit superior is not empty. By contrast, $x \in \lim_m \inf_{n\geq m} A^n$ if and only if

$$\exists m \in \mathbb{N}, \quad x \in \bigcap_{n\geq m} A^n,$$

indicating that there is an integer m such that $x \in A^n$ for each $n \geq m$, that is, $x \notin A^n$ for only finitely many n.

The limit inferior may be empty; it is not empty if all A^n after a certain period m approve some option. In short, the limit inferior consists of elements that "eventually stay forever" (are in each set after some m), while the limit superior consists of elements that "never leave forever" (are in some set after each m). Note that

$$\lim_m \inf_{n\geq m} A^n \subseteq \lim_m \sup_{n\geq m} A^n$$

and if $\lim_m \inf_{n\geq m} A^n = \lim_m \sup_{n\geq m} A^n$, then it easy to verify that $(A^n)_{n\in\mathbb{N}}$ is a convergent sequence.[11]

The use of limit concepts is the key to extending the approval voting method. A naive approach is to take individual ballots B_i as elements in the sequence, that is, $A^n = B_n$. Under this approach, each convergent sequence of ballots has some m such that all generations born after period m cast the same ballot, and thus, a consensus among the future generations approximately exists. In the ballot setting, an option in the limit superior is approved by infinitely many future generations, while one in the limit inferior is approved by all generations after a certain period. Notice that each of the three limit concepts represents agreements of future generations in a certain sense. However, this approach cannot plausibly resolve intergenerational conflicts as long as the approval voting method is taken as an acceptable collective decision process for a finite population. For instance, consider a profile such that generations before $m > 0$ approve only x^1 and those after m approve not only x^1 but x^2. The limit suggests that x^1 and x^2 are chosen, but it ignores the interests of the

[11] In the literature on intergenerational equity, the "lim inf" criterion is employed by Fleurbaey and Michel (2003) and Chambers (2009). In their works, it is used to compare infinite streams of outcomes/utilities. Here, we follow this strategy to compare infinite streams of ballots.

present generations and generations in the near future. Indeed, for any point in time t, approval voting among generation t and generations before t suggests that only x^1 must be chosen. To incorporate their interests, we use a sequence of outcomes of finite-population approval voting explicitly. For each period n, let $M_{\mathbf{B}}^n$ be the outcome of the finite-population approval voting among generations from 1 to n. Then, the following sequence is given:

$$M_{\mathbf{B}}^1, M_{\mathbf{B}}^2, M_{\mathbf{B}}^3, \ldots, M_{\mathbf{B}}^n, \ldots$$

For the above-mentioned ballot profile, the sequence is completely constant, that is, $M_{\mathbf{B}}^n = \{x^1\}$. We propose to use the above three limit concepts for this sequence $(M_{\mathbf{B}}^n)$ of outcomes. For a large number $m > 0$, $M_{\mathbf{B}}^m$ represents a resolved conflict from a long-term perspective. Thus, it is plausible to examine its limit, limit superior, or limit inferior.

An outcome $x^* \in X$ is said to be **stationarily approved** for $\mathbf{B} \in \mathcal{B}^{\mathbb{N}}$ if

$$\exists m \in \mathbb{N} : \forall n \geq m, \quad x^* \in M_{\mathbf{B}}^n,$$

where, similarly as before, $M_{\mathbf{B}}^n$ is the set of outcomes that receive the greatest number of approvals up to n in $\mathbf{B}$. That is, x^* is stationarily approved if and only if it eventually stays forever in the sets of outcomes that receive the greatest number of approvals, meaning that

$$x^* \in \lim_m \inf_{n \geq m} M_{\mathbf{B}}^n.$$

We note that our definition of stationarily approved outcome is closely related to the **overtaking criterion**, which is widely employed in the literature.[12] Interestingly, axiomatics for the overtaking criterion are substantially different from that for the "lim inf" criterion in the literature on intergenerational equity,[13] although the two criteria are peacefully harmonized in our infinite ballot setting.

An outcome $x^* \in X$ is said to be **stationarily disapproved** for $\mathbf{B} \in \mathcal{B}^{\mathbb{N}}$ if

$$\exists m \in \mathbb{N} : \forall n \geq m, \quad x^* \notin M_{\mathbf{B}}^n.$$

Thus, x^* is stationarily disapproved if and only if it eventually stays forever outside the sets of outcomes which receives the greatest number of approvals. As the following example shows, it can be the case that an outcome is neither stationarily approved nor stationarily disapproved. Note also that x^* is **not** stationarily disapproved for $\mathbf{B} \in \mathcal{B}^{\mathbb{N}}$ if and only if

$$\forall m \in \mathbb{N} : \exists n \geq m, \quad x^* \in M_{\mathbf{B}}^n,$$

[12] The overtaking criterion is defined as follows: given two infinite utility streams $(u_i)_{\in \mathbb{N}}, (u'_i)_{i \in \mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$, $(u_i)_{\in \mathbb{N}}$ is at least as good as $(u'_i)_{\in \mathbb{N}}$ if there exists $m \in \mathbb{N}$ such that $\sum_{i=1}^n u_i \geq \sum_{i=1}^n u'_i$ for each $n \geq m$.

[13] See Fleurbaey and Michel (2003), who propose the "lim inf" criterion as follows: $(u_i)_{\in \mathbb{N}}$ is at least as good as $(u'_i)_{\in \mathbb{N}}$ if $\lim_m \inf_{n \geq m} \sum_{i=1}^n (u_i - u'_i) \geq 0$. They argue the difference between the "lim inf" criterion and the overtaking criterion in their framework.

that is, if and only if x^* never leaves forever the sets of outcomes that receive the greatest number of approvals. In short, x^* is not stationarily disapproved if and only if

$$x^* \in \lim_m \sup_{n\geq m} M^n_{\mathbf{B}}.$$

Example 7 Assume that $X = \{x^1, x^2, x^3\}$, and consider the profile $\mathbf{B} \in \mathcal{B}^{\mathbb{N}}$ in which $B_i = \{x^1\}$ for each voter $i \in \{1, 3, 5, 7, 9, \ldots\}$ and $B_i = \{x^2\}$ for each voter $i \in \{2, 4, 6, 8, 10, \ldots\}$. We see that, for each $n \geq 1$, $x^1 \in M^n_{\mathbf{B}}$. This means that x^1 is a stationarily approved outcome. On the contrary, x^3 is never chosen by a generation. It follows that, for each $n \geq 1$, $x^3 \notin M^n_{\mathbf{B}}$ and so, by definition, x^3 is a stationarily disapproved outcome. In this example, x^2 is neither stationarily approved nor stationarily disapproved. Indeed, for each odd $n \in \{1, 3, 5, \ldots\}$, $x^2 \notin M^n_{\mathbf{B}}$. This implies that there is no $m \in \mathbb{N}$ such that, for each $n \geq m$, $x^2 \in M^n_{\mathbf{B}}$. Therefore, x^2 is not a stationarily approved outcome. On the other hand, for each even $n \in \{2, 4, 6, \ldots\}$, $x^2 \in M^n_{\mathbf{B}}$. This implies that there is no $m \in \mathbb{N}$ such that, for each $n \geq m$, $x^2 \notin M^n_{\mathbf{B}}$. Therefore, x^2 is not a stationarily disapproved outcome. □

Let $\mathcal{S} \subseteq \mathcal{B}^{\mathbb{N}}$ be the subset of profiles of $\mathcal{B}^{\mathbb{N}}$ such that each outcome is either stationarily approved or stationarily disapproved. A profile in $\mathcal{S}$ is called **stationary**. From the above definitions, $\mathcal{S}$ is the subset of profiles of $\mathcal{B}^{\mathbb{N}}$ such that the sequence $(M^n_{\mathbf{B}})_{n\in\mathbb{N}}$ is convergent:

$$\lim_m \inf_{n\geq m} M^n_{\mathbf{B}} = \lim_m \sup_{n\geq m} M^n_{\mathbf{B}},$$

or, equivalently,

$$\exists m \in \mathbb{N} : \forall n \geq m, \quad M^n_{\mathbf{B}} = M_{\mathbf{B}} \text{ for some } M_{\mathbf{B}} \subseteq X. \tag{8}$$

The set $M_{\mathbf{B}}$ is the **set of stationarily approved outcomes** of $\mathbf{B}$. In particular, any profile in Ω belongs to $\mathcal{S}$ and a profile of the type $(A, A, \ldots)$, where $\emptyset \neq A \subset X$, belongs to $\mathcal{S} \setminus \Omega$.

Example 8 Assume again that $X = \{x^1, x^2, x^3\}$ and consider the profile $\mathbf{B}$ in which $B_i = \{x^1\}$ for each voter $i \in \{1, 2, 4, 6, 8, 10, \ldots\}$, and $B_i = \{x^2\}$ for each voter $i \in \{3, 5, 7, 9, 11, \ldots\}$. In this profile, x^1 is stationarily approved, whereas the other outcomes are stationarily disapproved. Thus, this profile is an element of $\mathcal{S}$. □

For any profile in $\mathcal{S}$, it is natural to require that a stationarily approved outcome is chosen and a disapproved one is not. This leads us to the following definition.

A generalized ballot aggregation function f on the domain $\mathcal{B}^{\mathbb{N}}$ **respects stationary approval** if for each $\mathbf{B} \in \mathcal{S}$, $f(\mathbf{B})$ coincides with the set of stationary approved outcomes $M_{\mathbf{B}}$. Because $\mathbf{B} \in \mathcal{S}$, we have

$$f(\mathbf{B}) = M_{\mathbf{B}} = \lim_n M^n_{\mathbf{B}}.$$

The class of generalized ballot aggregation functions that respect stationary approval is denoted by $\mathcal{F}_S$.

Note that the above requirement does not impose any restriction on a profile not in $\mathcal{S}$. For instance, the ballot profile provided in Example 7 does not belong to $\mathcal{S}$. Nevertheless, it seems natural to expect that x^1 is chosen and x^3 is not. Example 7 leads us to the following stronger notion of approval voting where f selects the subset of stationarily approved outcomes and excludes the subset of outcomes that are stationarily disapproved. Formally, f **strictly respects stationary approval** if

$$\forall \mathbf{B} \in \mathcal{B}^{\mathbb{N}}, \quad \lim_m \inf_{n \geq m} M^n_{\mathbf{B}} \subseteq f(\mathbf{B}) \subseteq \lim_m \sup_{n \geq m} M^n_{\mathbf{B}}.$$

The class of generalized ballot aggregation functions that strictly respect stationary approval is denoted by $\mathcal{F}^*_S$. It must be clear that if f strictly respects stationary approval, then it respects stationary approval. Indeed, on the subdomain $\mathcal{S}$, $(M^n_{\mathbf{B}})_{n\in\mathbb{N}}$ is a convergent sequence so that $\lim_m \inf_{n\geq m} M^n_{\mathbf{B}} = \lim_m \sup_{n\geq m} M^n_{\mathbf{B}} = \lim_n M^n_{\mathbf{B}}$, and so $f(\mathbf{B}) = \lim_n M^n_{\mathbf{B}}$. We remark that this strict notion of stationary approval does not require any specific treatment of non-approved and non-disapproved outcomes in our sense, for instance, the outcome x^2 in Example 7, which is neither stationarily approved nor stationarily disapproved. Therefore, some generalized ballot aggregation functions in $\mathcal{F}^*_S$ choose x^2 in Example 7, while others do not.

With respect to set inclusion, one distinguishable generalized ballot aggregation function is the largest among the generalized ballot aggregation functions in $\mathcal{F}^*_S$. That is, it coincides with the subset of outcomes that are not stationarily disapproved. Formally, the **infinite approval voting** f^{IA} is defined as follows:

$$\forall \mathbf{B} \in \mathcal{B}^{\mathbb{N}}, \quad f^{IA}(\mathbf{B}) = \lim_m \sup_{n \geq m} M^n_{\mathbf{B}}.$$

To sum up, we have the following relationships:

$$f^{IA} \in \mathcal{F}^*_S \subseteq \mathcal{F}_S.$$

Example 9 Assume that $X = \{x^1, x^2, x^3\}$, and consider a profile $\mathbf{B}$ in which $B_i = \{x^1\}$ for each voter $i \in \{1, 4, 5, 8, 9, \ldots\}$ and $B_i = \{x^2\}$ for each $i \in \{2, 3, 6, 7, 10, 11, \ldots\}$. Here, neither x^1 nor x^2 is stationarily approved, while x^3 is stationarily disapproved. We have $f^{IA}(\mathbf{B}) = \{x^1, x^2\}$. □

We now introduce key axioms of continuity for characterizing the above three classes of generalized ballot aggregation functions that are consistent with the approval voting function in the finite-population setting. Before providing the definitions, we need a notation. For each $\mathbf{B} = (B_i)_{i\in N} \in \mathcal{B}^{\mathbb{N}}$ and each $n \in \mathbb{N}$, we define the associated profile $\mathbf{B}^n \in \Omega$ whose support I_n belongs to $[\![1, n]\!]$ and is constructed from the first n ballots of $\mathbf{B}$, to which, we add an infinite number of ballots cast by unconcerned voters. Formally,

$$\mathbf{B}^n = (B_1, \ldots, B_n, \mathbf{X}) \in \Omega.$$

The first axiom of continuity indicates that for each profile $\mathbf{B}$, if the sequence formed by the aggregation $f(\mathbf{B}^n)$ with $n \in \mathbb{N}$ converges, then $f(\mathbf{B})$ must select its limit. Given a generalized ballot aggregation function f, we denote by $\mathcal{S}^f$ the set of profiles $\mathbf{B} \in \mathcal{B}^{\mathbb{N}}$ such that the sequence $(f(\mathbf{B}^n))_{n\in\mathbb{N}}$ is convergent.

Choice continuity: It holds that

$$\forall \mathbf{B} \in \mathcal{S}^f, \quad f(\mathbf{B}) = \lim_n f(\mathbf{B}^n).$$

This axiom is closely related to Axiom 3 of Brock (1970), which is employed to analyze the overtaking criterion. According to him, it "captures the notion that decisions on infinite programs are consistent with decisions on finite programs of length n if n is large enough" (Brock 1970, p.929). Since his framework entirely depends on preference relations, choice continuity can be regarded as an extension of Brock's Axiom 3 to a choice-function setting.[14]

If the sequence formed by the aggregation $f(\mathbf{B}^n)$ with $n \in \mathbb{N}$ is not convergent, choice continuity is silent. The next axiom indicates that, in such a case, $f(\mathbf{B})$ contains the limit inferior of this sequence and is a subset of its limit superior.

Strong choice continuity: For each $\mathbf{B} \in \mathcal{B}^{\mathbb{N}}$, it holds that

$$\lim_m \inf_{n\geq m} f(\mathbf{B}^n) \subseteq f(\mathbf{B}) \subseteq \lim_m \sup_{n\geq m} f(\mathbf{B}^n).$$

Of course, strong choice continuity implies choice continuity because if $(f(\mathbf{B}^n))_{n\in\mathbb{N}}$ is a convergent sequence, then its limit inferior coincides with its limit superior. The following is a further refinement of strong choice continuity.

Full choice continuity: For each $\mathbf{B} \in \mathcal{B}^{\mathbb{N}}$, it holds that

$$f(\mathbf{B}) = \lim_m \sup_{n\geq m} f(\mathbf{B}^n).$$

The following result provides comparable axiomatic characterizations of $\mathcal{F}_S$, $\mathcal{F}_S^*$, and f^{IA}.

Theorem 3 *Let f be a generalized ballot aggregation function on the domain $\mathcal{B}^{\mathbb{N}}$. We have the following characterization results:*

(i) f satisfies finite faithfulness, finite consistency, finite cancellation, and choice continuity if and only if $f \in \mathcal{F}_S$;
(ii) f satisfies finite faithfulness, finite consistency, finite cancellation, and strong choice continuity if and only if $f \in \mathcal{F}_S^*$;

[14] See Asheim and Tungodden (2004) and Basu and Mitra (2007). They introduce variants of Brock's axiom. In particular, the formulation of axioms by Asheim and Tungodden (2004) is closely related to our axioms because they employ expressions with the lim inf and lim sup. A substantial difference is that they define their continuity axioms for binary relations, while we define our continuity axioms for choice functions.

(iii) f satisfies finite faithfulness, finite consistency, finite cancellation, and full choice continuity if and only if $f = f^{IA}$.

Proof First, notice that for each $\mathbf{B} \in \mathcal{B}^{\mathbb{N}}$, and each $n \in \mathbb{N}$, we have $\mathbf{B}^n \in \Omega \subseteq \mathcal{S}$. Thus, the set $M_{\mathbf{B}^n}$ of stationarily approved candidates of $\mathbf{B}^n$ is well-defined. Furthermore, by definition of $M_{\mathbf{B}}^n$, $M_{\mathbf{B}^n}^{I_n}$ and $M_{\mathbf{B}^n}$, it holds that

$$M_{\mathbf{B}}^n = M_{\mathbf{B}^n}^{I_n} = M_{\mathbf{B}^n},$$

where I_n is the support of $\mathbf{B}^n$. Next, if f satisfies finite faithfulness, finite consistency, and finite cancellation, then, by Theorem 2, for each $\mathbf{B} \in \mathcal{B}^{\mathbb{N}}$ and each $n \in \mathbb{N}$, we have: $f(\mathbf{B}^n) = M_{\mathbf{B}^n}^{I_n}$, which yields

$$M_{\mathbf{B}}^n = f(\mathbf{B}^n). \tag{9}$$

Thus, the sequences $(M_{\mathbf{B}}^n)_n$ and $(f(\mathbf{B}^n))_n$ are equal, which implies that $\mathcal{S} = \mathcal{S}^f$. Therefore, under the hypothesis that f satisfies finite faithfulness, finite consistency, and finite cancellation, we have the following equivalences:

(i) [f satisfies choice continuity] $\iff$ [f respects stationary approval].
Indeed, if f satisfies choice continuity, for $\mathbf{B} \in \mathcal{S} = \mathcal{S}^f$, we have the following:

$$f(\mathbf{B}) = \lim_n f(\mathbf{B}^n) = \lim_n M_{\mathbf{B}}^n = M_{\mathbf{B}},$$

where the first equality comes from choice continuity, the second equality comes from (9) and the third equality is a consequence of the fact that $\mathbf{B} \in \mathcal{S}$. Thus, $f(\mathbf{B})$ selects $M_{\mathbf{B}}$, which proves that f respects stationary approval. Conversely, if f respects stationary approval, for $\mathbf{B} \in \mathcal{S}^f = \mathcal{S}$, we have the following:

$$\lim_n f(\mathbf{B}^n) = \lim_n M_{\mathbf{B}}^n = M_{\mathbf{B}} = f(\mathbf{B}),$$

where the first equality comes from (9), the second equality comes from the fact that $\mathbf{B}$ belongs to $\mathcal{S}$, and the third equality is a consequence of the fact that f respects stationary approval.

(ii) [f satisfies strong choice continuity] $\iff$ [f strictly respects stationary approval].
Indeed, if f satisfies strong choice continuity, then $\mathbf{B} \in \mathcal{B}^{\mathbb{N}}$, it holds that

$$\lim_m \inf_{n \geq m} f(\mathbf{B}^n) \subseteq f(\mathbf{B}) \subseteq \lim_m \sup_{n \geq m} f(\mathbf{B}^n) \tag{10}$$

By (9), expression (10) can be rewritten as:

$$\lim_m \inf_{n \geq m} M_{\mathbf{B}}^n \subseteq f(\mathbf{B}) \subseteq \lim_m \sup_{n \geq m} M_{\mathbf{B}}^n, \tag{11}$$

which shows that f strictly respects stationary approval. Conversely, if f strictly respects stationary approval, then (11) holds. By (9), expression (11) can be rewritten as (10), which means that f satisfies strong choice continuity.

(iii) [f satisfies full choice continuity] $\iff$ [f is the infinite approval voting]. The proof is similar to the case (i). It suffices to replace lim by lim sup.

This provides direct implications of Theorem 3. Conversely, assume that $f \in \mathcal{F}_S$. Thus, let $\mathbf{B} \in \Omega$ whose support belongs to $[\![1, n]\!]$, $n \in \mathbb{N}$. It follows that $\mathbf{B} \in \mathcal{S}$. Because f respects stationary approval, $f(\mathbf{B})$ coincides with the set of stationarily approved candidates $M_{\mathbf{B}}$ under $\mathbf{B}$ given by (8). Observe that $M_{\mathbf{B}} = M_{\mathbf{B}}^n$ so that $f(\mathbf{B}) = M_{\mathbf{B}}^n$. By Theorem 2, f satisfies finite faithfulness, finite consistency and finite cancellation. Furthermore, the above equivalences (i), (ii), and (iii) apply, from which we conclude that if f (strictly) respects stationary approval, then it satisfies (strong) choice continuity, and if $f = f^{IA}$, then it satisfies full choice continuity. □

We now explain the implications of Theorem 3. First, these results demonstrate the existence of plausible extensions of approval voting in the infinite-population setting; that is, there exists f that satisfies all proposed axioms. Second, continuity properties of choices are crucial for these extensions. Each of the three extensions is obtained by adding one continuity axiom to the three basic axioms. A stronger continuity yields a more restricted class of ballot aggregation functions. In particular, full choice continuity is strong enough for pining down only one function, the infinite approval voting.

The fact that an exact characterization is obtained for the infinite approval voting does not imply that the method is the most desirable among the voting methods in $\mathcal{F}_S$ (or $\mathcal{F}_S^*$). A distinguishable feature of the infinite approval voting method is its simplicity: it is always explicitly constructible by operating the limit superior to a sequence of outcomes of the finite-setting approval voting. However, the infinite approval voting method may include many outcomes. That is, if there is a conflict between two groups of future generations, as in Example 7, then the voting method does not resolve the conflict and approve what each group prefers. As a result, the infinite approval voting method selects an option only if this option never leaves forever the sets of outcomes that receive the greatest number of approvals, that is, if this option is not stationarily disapproved. This emerges from the nature of the limit superior.

The concept of the limit superior is related to the idea of admissibility by Levi (1986), who argues how difficult conflicts can be resolved. Options in the feasible set are admissible if they "have not been prohibited by the agent's value commitments from being chosen by the agent," and, thus, they are "admissible relative to the agent's valuations of the feasible options as better or worse, all things considered" (Levi (1986), p. 10). In our context, an option is in the limit superior if and only if it is not stationarily disapproved. This constitutes the process of generating a set of admissible options in the infinite-population setting of approval voting. Levi (1986) emphasizes that there must be some procedure to choose an optimal option among the admissible options. That is, admissible options are outcomes that survived after excluding "unchoosable" options, but all admissible options do not have to be chosen.

Based on this understanding, infinite approval voting is an aggregation rule that always chooses all admissible options. This is the reason why it chooses many options.

Levi's idea also conveys that a certain subset of the limit superior must be chosen as optimal options. These rules are elements of $\mathcal{F}_S$ (or $\mathcal{F}_S^*$). Put differently, there can be a more plausible aggregation rule in $\mathcal{F}_S$ (or $\mathcal{F}_S^*$). This means that weaker continuity axioms are more reasonable as axioms for considering optimal options. Full continuity is meaningful even in this line of thoughts; it is considered to be a natural axiom for investigating admissible options.

Our question then becomes as follows: which is the most plausible voting rule? That is, how can we resolve indeterminacy or incompleteness under our infinite-population setting? This is the choice among voting rules in $\mathcal{F}_S$ (or $\mathcal{F}_S^*$). Under the standard axiological approach, some new axiom is imposed in addition to the three basic axioms and choice continuity (or strong choice continuity) for answering this question. However, a suspension is required. In a study that examines Levi's argument, Sen (2004, p. 55) claims that all conflicts do not have to be resolved; Sen called this "assertive incompleteness". Indeed, why intergenerational conflicts in the current setting may not be completely resolved is due to several reasons. First, the relevance of our setting is dependent on the presence of difficulties for accounting utilities of the future generations. That is, it is quite difficult to obtain information about the utilities of the future generations and, also, intergenerational utility comparison is also an issue. These difficulties suggest that our infinite-population extensions of approval voting are built on a non-ideal informational circumstance, and thus, tentative incompleteness is deeply included.[15]

Second, our setting contains infinitely many generations. It has been demonstrated that a quite standard axiology of utilitarianism can break down in the presence of an infinite population; see Van Liedekerke (1995) and Van Liedekerke and Lauwers (1997). A similar problem can occur under our setting: impossibilities easily occur for the time-neutral class of rules.

To illustrate this, consider the following profile:

$$B_i = \{x^1\} \text{ for } i \in \{1, 2, 4, 6, 8, \dots\} \text{ and } B_i = \{x^2\} \text{ for } i \in \{3, 5, 7, 9, \dots\}.$$

It seems that a normatively reasonable choice is $\{x^1\}$ because it is the only one that is stationarily approved. Now a consider the following permutation $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ on the set of generations:

$$\sigma(1) = 2, \text{ and } \sigma(2i) = 2i + 2, \ \sigma(2i + 1) = 2i - 1 \ \text{ for } i \in \mathbb{N}.$$

If the permutation σ is applied on the set of generations, we obtain a new profile $\mathbf{B}'$ such that $B'_{\sigma(1)} = B_1$, $B'_{\sigma(2i)} = B_{2i}$, and $B'_{\sigma(2i+1)} = B_{2i+1}$ for each generation $i \in \mathbb{N}$, that is,

$$B'_i = \{x^1\} \text{ for } i \in \{2, 4, 6, 8, \dots\} \text{ and } B'_i = \{x^2\} \text{ for } i \in \{1, 3, 5, 7, 9, \dots\}.$$

15 A natural interpretation of strong choice continuity is associated with tentative incompleteness. Choice continuity does not exclude unnecessary tentative incompleteness shown in Example 5. Strong choice continuity works as an axiom for resolving such incompleteness.

Now, x^2 is stationarily approved. This implies that our classes of voting rules are not invariant under a full permutation of the labels of generations. In other words, we cannot expect Full anonymity to be satisfied even if this axiom is natural under a finite-population setting.[16] Note that this problem does not occur in the domain of Theorem 2. Therefore, this non-invariance property is a consequence of our continuity axioms.

Moreover, there can be a difficulty in explicit construction of a method for choosing an optimal option among the admissible options. In the presence of an infinite population, a gap can exists between the existence of a rule and its constructibility. Indeed, it is easy to show the existence of a single-outcome rule by utilizing the Axiom of Choice, which allows us to choose one option for each profile. However, since there are infinitely many profiles, its constructibility is not guaranteed. It may be the case that the Axiom of Choice is necessary.[17] That is, it may be impossible to construct rules explicitly.

These points suggest that assertive incompleteness is potentially involved in our problem. That is, even if we have information about future generations and overcome the issue of comparability, choosing a single option is not necessary. In sum, resolving all incomparable factors are unnecessary, and thus, identifying a plausible subclass of voting rules is enough under our setting.

9 Concluding remarks

In this paper, we attempted to develop an infinite-population approval voting. Our results suggest that our framework of infinite-population approval voting is a plausible one for considering intergenerational conflicts and that the existing results in the finite-population case are naturally extended. According to our analysis, it is technically important that an operator for combining profiles is appropriately introduced.[18] Given such an operation, reasonable voting methods can be examined with an infinite population. Specifically, our proposal is to extend the approval ordering by applying the catching-up and overtaking approaches where interests of the future generations are naturally incorporated and well respected in these criteria.

We provide remarks on several directions for extending our analysis. First, our approach can be applied to a broader class of voting methods. Many voting rules

[16] Roughly speaking, odd generations appear earlier than even generations. Notably, the rule respects the interests of the earlier generations in a sense. This non-invariant nature is associated with what we may call the present bias. However, this bias occurs in limited cases. If one tries to avoid this bias and impose full anonymity, decision-making is either limited to profiles in Ω or another voting rule must be considered.

[17] See Litak (2018) for the use of the Axiom of Choice in a voting model with an infinite population.

[18] We now mention a slightly technical point on the relationship between finite-population and infinite-population settings. Our characterization results extend the existing works on a finite but "variable-population" case. A newly introduced operator $\oplus$ is needed for extensions to the infinite-population setting because of the variable nature of consistency. For approval voting methods or scoring methods, variable-population settings are considered to be quite natural; see Fishburn (1978a,1978b) and Young (1974). Further, plausible features of these voting methods are captured with variable population axioms. However, by using the result in a fixed-population setting, our extensions can be simpler. Indeed, Baigent and Xu (1991) propose a set of axioms for characterizing approval voting under a finite- and fixed-population setting. If we use variations of their axioms, our extensions can be done without the operator $\oplus$.

have been proposed in the finite-population framework; we can extend them to the intergenerational-ballot model with an infinite population. In general, our three extension approaches can be applied to any reasonable class of voting methods. A second possible extension of our analysis comprises including multiple voters in each generation. In our analysis, each generation is treated as a single voter because the focus of this study was to consider an intergenerational conflict. However, for most practical issues, various conflicts of interest are present within a generation. Therefore, incorporating several agents in each generation is a natural direction for extending our result. Although it is not a trivial extension, we believe that our results and axioms are helpful; a more extended version of infinite-population approval voting will be obtained with some modification of our axioms. For this direction, the problem is how we can treat different population sizes among generations; see Kamaga (2016) for a work that explicitly includes this issue with utilitarian considerations.

Moreover, our methods can be useful for extending a judgment aggregation problem to an infinite-population case. That is, we consider the situation where people casts their ballots. The recent development of judgment aggregation shows that a general class of language can be considered in social aggregation problems; see List and Pettit (2002, 2004) and List (2012). Since most works on judgment aggregation focuses on the case with a finite population, a substantial development can be done by examining methods for extending it to the case with an infinite population. This is another remaining issue.

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Declarations

Conflict of interest None.

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